Armed Services Technical Information Agency

AD 30606

Reproduced by

DOCUMENT SERVICE CENTER

KNOTT BUILDING, DAYTON, 2, OHIO

UNCLASSIFIED
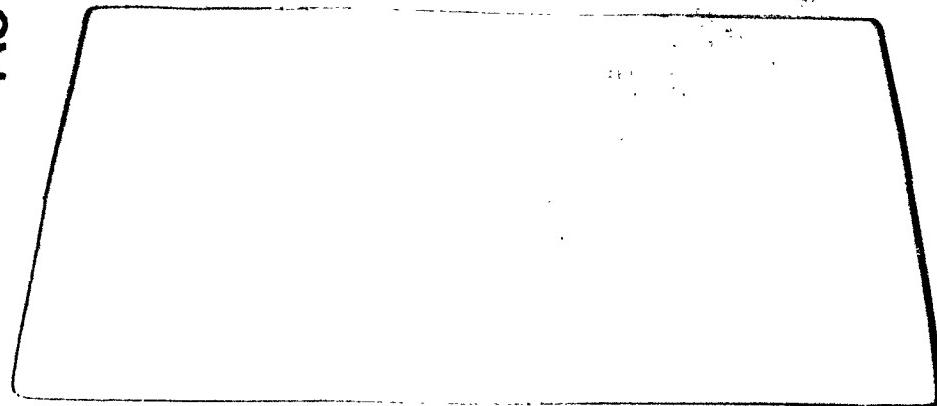
INSTITUTE OF INDUSTRIAL RESEARCH

DEPARTMENT OF ELECTRICAL ENGINEERING

Syracuse University

Syracuse 10, New York

ON THE CYLINDRICAL ANTENNA

R. F. Harrington

March 15, 1954
Contract Nonr 1198(00)
Placed by the Office of Naval Research

Institute of Industrial Research
Department of Electrical Engineering
Syracuse University
Syracuse 10, New York

CONTENTS

I. INTRODUCTION

The problem of the cylindrical antenna of appreciable diameter has received considerable attention from a number of investigators. Three approaches have been used in the literature. These are (a) Hallén's asymptotic solution to the approximate integral equation,[1] and King's modification of it,[2] (b) Storer's variational solution to the approximate integral equation,[3] and (c) Schelkunoff's approximate modal solution.[4] This report re-formulates the problem in terms of a mathematical model for which an exact solution is possible. The method of solution is outlined. A solution using Rumsey's reaction concept,[5] which is a straight forward approach to the variational formulation, is also given.

Interest is restricted to steady-state, harmonic time-varying fields. All time-varying quantities are represented in terms of sinors.[6] For a scalar quantity,

$$v = \sqrt{2}\,|V|\,\sin(\omega t + \alpha) = \mathrm{Im}(\sqrt{2}\,V\,e^{j\omega t}) \,, \qquad (I\text{-}1)$$

where v is a time-varying scalar and $V = |V|e^{j\alpha}$ is the sinor of v. For a vector quantity,

$$\underline{\mathcal{E}} = \mathrm{Im}(\sqrt{2}\,\underline{E}\,e^{j\omega t}) \,, \qquad (I\text{-}2)$$

where $\underline{\mathcal{E}}$ is a time-varying vector and $\underline{E}$ is the sinor of $\underline{\mathcal{E}}$. Eq. (I-2) should be interpreted to mean that the components of $\underline{\mathcal{E}}$ are related to the components of $\underline{E}$ according to Eq. (I-1).

The electromagnetic field is represented in terms of two vectors, the electric intensity and the magnetic intensity, the sinors of which are $\underline{E}$ and $\underline{H}$, respectively. The characteristics of a region are expressed in terms of the parameters σ, ϵ, and μ, called the conductivity, capacitivity, and inductivity, respectively. Sources are represented in terms of impressed electric

current density and impressed magnetic current density, the sinors of which are denoted by $\hat{\underline{J}}$ and $\hat{\underline{M}}$, respectively. The field equations in terms of sinors, extended to include impressed currents, are

$$\nabla \times \underline{H} = (j\omega\epsilon + \sigma)\underline{E} + \hat{\underline{J}} \ , \tag{I-3}$$

$$-\nabla \times \underline{E} = j\omega\mu \, \underline{H} + \hat{\underline{M}}. \tag{I-4}$$

The right hand side of Eq. (I-3) is the sinor of the total electric current density, $\hat{\underline{J}}^t$, and the right hand side of Eq. (I-4) is the sinor of the total magnetic current density, $\hat{\underline{M}}^t$. In integral form, the field equations are

$$\oint_C \underline{H} \cdot d\underline{l} = \iint_S \hat{\underline{J}}^t \cdot d\underline{s} = I^t \ , \tag{I-5}$$

$$-\oint_C \underline{E} \cdot d\underline{l} = \iint_S \hat{\underline{M}}^t \cdot d\underline{s} = K^t \ , \tag{I-6}$$

where I^t is the sinor of the total electric current through the surface S encircled by the contour C, and K^t is the sinor of the total magnetic current. The voltage is defined to be the line integral of the electric intensity. Thus, the relation

$$V = \int_C \underline{E} \cdot d\underline{l} \ , \tag{I-7}$$

gives the sinor of a voltage between the "terminals" of the curve C. The value of V is substantially independent of all more or less direct paths if taken in a source-free region and if the terminals are close together. The conduction current is defined to be the surface integral of the conduction current density. Thus, the relation

$$I = \iint_S \sigma \underline{E} \cdot d\underline{s} \ , \tag{I-8}$$

gives the sinor of the conduction current. The value of I in a "wire" is substantially independent of all more or less tightly stretched surfaces if

the cross section is small. The impedance Z between two wires close together
is defined to be the ratio V/I, where V is the sinor of the voltage between
the wires and I is the current in the wires.

As a mathematical aid, it is convenient to introduce auxiliary functions,
called vector potentials, in terms of which the field can be expressed. For a
linear, homogeneous media, if $\hat{M} = 0$, one can write

$$\underline{H} = \nabla \times \underline{A} , \tag{I-9}$$

where $\underline{A}$ is called a magnetic vector potential. Note that $\underline{A}$ is not unique,
but is determined only to within the gradient of a scalar since $\nabla \times \nabla\phi = 0$.
One can also use an electric vector potential for regions in which $\hat{J} = 0$, but
this is not necessary in this report. From the field equations, $\underline{E}$ in terms of
$\underline{A}$ is given by

$$(j\omega\epsilon + \sigma)\underline{E} = \nabla \times \nabla \times \underline{A} . \tag{I-10}$$

A unique solution for $\underline{A}$ in terms of impressed electric currents in a region
homogeneous everywhere is the potential integral solution.[7] A solution
appropriate to cylindrical coordinates is given by $A_z = \psi$, where ψ is a solution
to the scalar wave equation.[8]

A powerful method of reducing electromagnetic problems to simpler mathe-
matical models is the equivalence principle.[9] In the literature, the same
results are usually obtained through the use of Green's functions, but the
equivalence principle is usually more direct. A concise statement of this
principle is as follows. Consider a surface S enclosing the sources of an
electromagnetic field, $\underline{E},\underline{H}$. Now consider a mathematical model consisting of
the same surface S and with the same matter (σ, ϵ, μ the same) external to S.
On S are impressed the surface currents

$$\underline{J} = \underline{n} \times \underline{H}, \tag{I-11}$$

$$\underline{M} = \underline{E} \times \underline{n}, \tag{I-12}$$

with the source in the original problem removed. One now has the same field $\underline{E},\underline{H}$ external to S and zero field internal to S. Since there is no field within S, one can place any material desired internal to S without changing the field external to S. Particularly useful choices of matter to place within S are (a) free space, (b) a perfect electric conductor, and (c) a perfect magnetic conductor. For case (a), one can use the potential integral solution.[7] For case (b), it can be shown that the electric currents produce no field, giving a representation in terms of only $\underline{M}$. For case (c), the magnetic currents produce no field, giving a representation in terms of only $\underline{J}$.

In this report, use is made of the theory of generalized Fourier transforms.[10] A transform pair is given formally by

$$\bar{F}(w) = \int_{-\infty}^{\infty} F(z) \, e^{-jwz} dz, \tag{I-13}$$

$$F(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{F}(w) \, e^{jwz} \, dw. \tag{I-14}$$

Eq. (I-13) is called the transformation integral, and Eq. (I-14) is called the inverse transformation integral. The existence of $\bar{F}(w)$ implies certain restrictions on $F(z)$. It can be shown that the transforms of the field vectors exist if the sources can be contained in a finite volume.[11] A theorem which finds application in this report is Parseval's theorem for transforms. This is

$$\int_{-\infty}^{\infty} F(z) \, G(z) \, dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{F}(w) \, \bar{G}(-w) \, dw. \tag{I-15}$$

As defined by Rumsey,[5] the reaction between two sets of sources, $\hat{\underline{J}}^a, \hat{\underline{M}}^a$ and $\hat{\underline{J}}^b, \hat{\underline{M}}^b$, is

$$\langle a,b \rangle = \iiint (\underline{E}^b \cdot \hat{\underline{J}}^a - \underline{H}^b \cdot \hat{\underline{M}}^a) \, dv, \tag{I-16}$$

where $\underline{E}^b, \underline{H}^b$ is the field produced by the b sources. If the sources can be contained in a finite volume and if the medium is linear, the reciprocity

theorem[12] states that

$$\langle a,b \rangle = \langle b,a \rangle . \qquad (I-17)$$

The self reaction of a source is that given by Eq. (I-16) for a = b. Note that the self-reaction of a circuit source is VI. If a set of sources are multiplied by some constant, the reaction is increased proportionately, that is

$$\langle Ka,b \rangle = K\langle a,b \rangle . \qquad (I-18)$$

Rumsey presents a method of formulating problems to obtain an approximate solution which is "best in the physical sense".[5] The result is essentially a stationary expression, insensitive to small variations in a current distribution about the correct distribution.

II. FORMULATION OF THE PROBLEM

The cylindrical antenna is considered to consist of two conducting rods, separated by a small gap, and excited by a generator connected across the gap. This is illustrated in Fig. 1. The generator is assumed to be physically small so that it can be contained within the cylindrical gap. For this report, the rods are taken to be perfect conductors.

The equivalence principle is now applied to the cylindrical surface just enclosing the antenna, including the gap generator. Choosing to back the equivalent currents by a perfect conductor, one has the problem exactly represented by a single conducting rod with a loop of impressed magnetic currents encircling what was formerly the gap. This is shown in Fig. 2. The self-reaction seen by the magnetic currents will be the same as that seen by the generator in the original problem.

Another representation of the problem is in terms of both the equivalent electric and magnetic currents over the cylinder, with both the conductor and
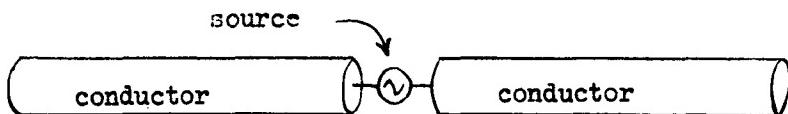
Fig. 1. The Cylindrical Antenna.
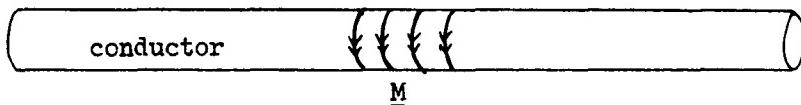
Fig. 2. The Equivalent Problem.
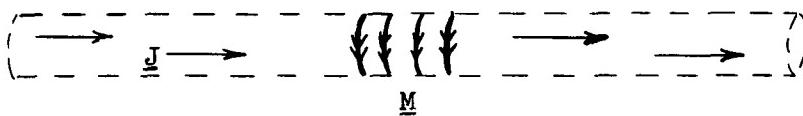
Fig. 3. The Free-Space Problem.
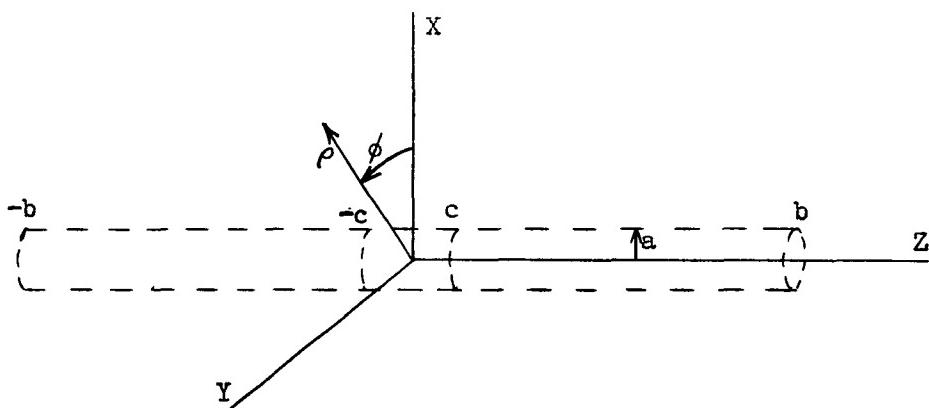
Fig. 4. Dimensions and Coordinates.

generator removed. This is called the free-space representation, and is shown in Fig. 3. The equivalent electric currents of the free-space representation are equal to the conduction currents on the rod of Fig. 2.

For the mathematical analysis, the coordinate system and dimensions shown in Fig. 4 are used. The coordinates are the usual ρ, ϕ,z circular cylindrical ones. The total length of the antenna is 2b, its diameter is 2a, and the length of the gap is 2c. Only antennas symmetrical with respect to the plane z = 0 are considered.

It is noted that the problem is symmetrical about the z axis. One can conclude therefrom that the impressed magnetic currents of the equivalent problems must flow only in the ϕ-direction. The conduction currents along the side of the cylinder of Fig. 2, and consequently the impressed electric currents of the free-space representation, must flow only in the z-direction. Over the ends of the cylinder, the electric currents must flow only in the ρ-direction. If the diameter of the cylinder is small, the effect of the currents on the ends of the cylinder is known to be small. There are two types of approximations that can be made because of the smallness of the "end effects". One is to formulate an approximate equation neglecting the end currents, as was done by Hallén. The other is to formulate exact equations for the approximate model obtained by considering the conductor of Fig. 2 to be hollow. As pointed out by Schelkunoff,[13] only the latter type of formulation can be expected to have an exact mathematical solution.

Primary interest is in antennas having a small gap. For this case, the exact distribution of the equivalent magnetic currents is unimportant, the solution being essentially the same for all well-behaved distributions. It is assumed that E_z in the gap of the original problem, and consequently M_ϕ in the equivalent problem, is a constant. Applying Eq. (I-6) to a path just

enclosing the magnetic currents, and making use of Eq. (I-7), it can be shown that

$$M_\phi(z) = \frac{-V}{2c} , \quad |z| < c. \tag{II-1}$$

This is taken to be the source in the mathematical problem. Of primary interest is the input impedance Z, or the input admittance Y, of the antenna. This is the impedance or admittance seen by the source. It can be expressed in the following ways:

$$Z = \frac{1}{Y} = \frac{V}{I} = \frac{\langle s,s \rangle}{I^2} = \frac{V^2}{\langle s,s \rangle} , \tag{II-2}$$

where $\langle s,s \rangle$ represents the self-reaction seen by the source. Since the source is expressed in terms of magnetic currents, the preferable form for a reaction solution is the last one on the right side of Eq. (II-2). If an exact solution is obtained, all forms are equivalent. There is also some interest in obtaining the current distribution along the conductor, since it is primarily this which determines the radiation pattern.

III. THE TRANSFORM EQUATION

Considering the conductor of Fig. 2 to be hollow, an exact transform equation can be obtained for the problem. From symmetry considerations, it follows that the field will be TM (transverse magnetic) to the z direction, and independent of ϕ. General formulas for a symmetric TM field from a cylinder of currents are derived in Appendix A. These involve the transforms of the field components in terms of the transforms of the electric and magnetic currents. The magnetic current is taken to be the known source, given by Eq. (II-1). The transform of $M_\phi(z)$ is calculated to be

$$\overline{M}_\phi(w) = - V \frac{\sin wc}{wc} . \tag{III-1}$$

Substituting this into the results of Appendix A, one has for the transform equation to the problem

$$\bar{E}(w) = F(w) + K(w)\,\bar{J}(w) , \tag{III-2}$$

where the known functions are

$$F(w) = \frac{V\pi a}{2j} \frac{\sin wc}{wc} \sqrt{\beta^2 - w^2}\; H_1^{(2)}\,(a\sqrt{\beta^2 - w^2}\,)\; J_0(a\sqrt{\beta^2 - w^2}\,) , \tag{III-3}$$

$$K(w) = \frac{-\pi a}{2wc}\,(\beta^2 - w^2)\; H_0^{(2)}\,(a\sqrt{\beta^2 - w^2}\,)\; J_0\,(a\sqrt{\beta^2 - w^2}\,) , \tag{III-4}$$

and the unknown transforms are

$$\bar{J}(w) = \int_{-b}^{b} J_z(z)\; e^{-jwz}\; dz, \tag{III-5}$$

$$\bar{E}(w) = \left(\int_{-\infty}^{-b} + \int_{b}^{\infty}\right)\, E_z(a,z)\; e^{-jwz} dz. \tag{III-6}$$

Eq. (III-2) is an exact equation for the mathematical model, and can be expected to have a precise solution. It can be shown[11] that if β is considered to be complex (corresponding to a dissipative media) all terms of Eq. (III-2) are regular in the strip $|\mathrm{Im}(w)| < |\mathrm{Im}(\beta)|$. As indicated by Eqs. (III-5) and (III-6), the inverse transforms of $\bar{J}(w)$ and $\bar{E}(w)$ exist over mutually exclusive intervals on z. The equation is similar to those previously solved by a modification of the Wiener-Hopf method of solution,[11] except for one important difference. The inverse transforms of the unknowns in Eq. (III-2) do not extend over semi-**infinite** intervals on z. For the benefit of anyone interested in attempting a solution of Eq. (III-2), the following is an outline of a procedure which shows some promise. Due to the complexity of the problem, the author has been unable to obtain an exact solution.

Suppose that $K(w)$ can be divided into the two functions

$$K(w) = K_1(w)/K_2(w) \tag{III-7}$$

such that

$$\int_{-\infty}^{\infty} \bar{E}(w) \, K_2(w) \, e^{jwz} \, dw = 0 \, , \quad |z| < b, \qquad (III-8)$$

$$\int_{-\infty}^{\infty} \bar{J}(w) \, K_1(w) \, e^{jwz} \, dw = 0 \, , \quad |z| > b. \qquad (III-9)$$

If K_1 and K_2 can be found, then an explicit expression for the solution can be written, obtained as follows. Eq. (III-2) can be put into the form

$$K_2(w) \, \bar{E}(w) - K_1(w) \, \bar{J}(w) = K_2(w) \, F(w) \, , \qquad (III-10)$$

where the inverse transform of the first term exists only over the interval $|z| > b$ and that of the second term exists only over the interval $|z| < b$. Transforming the inverse transform of Eq. (III-10) over the interval $|z| < b$, and performing the z integration then gives

$$\bar{J}(w) = \frac{-1}{\pi K_1(w)} \int_{-\infty}^{\infty} K_2(s) \, F(s) \, \frac{\sin(s-w)b}{s-w} \, ds. \qquad (III-11)$$

Since $J(w)$ is a finite transform, the current distribution in Fourier series form is given by the inversion formula.[10]

$$J_z(z) = \frac{1}{2b} \sum_{n=-\infty}^{\infty} \bar{J}(\frac{n\pi}{b}) \, e^{jn\pi z/b} \, , \quad |z| < b \, . \qquad (III-12)$$

This can be reduced to a cosine series since $J_z(z)$ is an even function. A solution such as outlined above would be quite complicated, but might simplify for small diameter antennas with narrow gaps.

Another method of solution which appears fruitful at first consideration is as follows. Suppose that one represents $J_z(z)$ by a Fourier series, Eq. (III-12), with the coefficients $\bar{J}(n\pi/b)$ to be determined. For the true solution, $E_z(a,z)$ is zero for $z < |b|$. For any other value of $J_z(z)$, a mean square error can be represented by

$$\frac{1}{2b} \int_{-b}^{b} E_z(a,z) \, E_z^*(a,z) \, dz = M \qquad (III-13)$$

For the true solution, M is zero, that is M is a minimum. Thus, M is to be reduced to a minimum through the choice of $J(n\pi/b)$. Setting $\partial M/\partial \bar{J}(n\pi/b) = 0$ gives equations for the determination of the $\bar{J}(n\pi/b)$. For an exact solution, there are an infinite number of coefficients to determine, giving an infinite number of simultaneous equations. This condition is what is referred to as "non-final determination of coefficients" by Sommerfield,[14] and is of no practical value. An approximate solution could be obtained by taking a finite number of terms in the series, or even taking some other functional approximation for $J_z(z)$, but the reaction approach presented in the next section appears to be better suited for this purpose.

IV. A REACTION SOLUTION

Consider the cylindrical antenna to be represented in terms of a conductor encircled by a loop of magnetic current, as shown in Fig. 2. The input admittance, from Eq. (II-2), is given by

$$Y = \frac{\langle s,s \rangle}{V^2} . \tag{IV-1}$$

Since the magnetic currents are directly proportional to V, it follows that $\langle s,s \rangle \sim V^2$, and Y is independent of the input voltage. Since the form of current distribution along the conductor is known approximately, one can expect a variational formulation of the problem to give accurate values of Y. The equation yielding the best answer can be derived in a straight forward manner by following Rumsey's reaction approach.[5]

In terms of the free-space representation of the problem, Fig. 3, the total reaction seen by the source is the free-space self-reaction of the magnetic currents, $\langle m,m \rangle$, plus the free-space reaction between the magnetic and electric currents, $\langle m,j \rangle$. That is,

$$\langle s,s \rangle = \langle m,m \rangle + \langle m,j \rangle . \tag{IV-2}$$

The term $\langle m,m \rangle$ involves only the magnetic currents and therefore depends only on the dimensions of the gap. The term $\langle m,j \rangle$ involves both the electric and magnetic currents, and thus depends on the dimensions of the gap and on the overall dimensions of the antenna, that is, on the dimensions of the conductor of the original problem, Fig. 1. Substituting Eq. (IV-2) into Eq. (IV-1), it is seen that the input admittance is the sum of two terms,

$$Y = Y_g + Y_c \; ,\qquad\qquad (IV-3)$$

where

$$Y_g = \frac{\langle m,m \rangle}{V^2} \; ,\qquad\qquad (IV-4)$$

$$Y_c = \frac{\langle m,j \rangle}{V^2} \; .\qquad\qquad (IV-5)$$

The term Y_g will be called the gap admittance, and the term Y_c will be called the conductor admittance. Since the magnetic current is taken to be the source, the Y_g can be calculated exactly. The problem is to determine the best approximation to Y_c. The quantity $1/Y_c$ is what most of the previous investigators have called the input impedance, although the necessity of including the effect of the gap has been recognized.

Along the cylinder coinciding with the original boundary of the antenna (just inside of the magnetic current), the tangential component of the electric intensity is zero. It follows that the tangential component of $\underline{E}$ along the cylinder due to the electric currents alone is just the negative of that due to the magnetic currents alone. Thus, even though the value of $\underline{J}$ is not known, the tangential component of $\underline{E}$ produced by $\underline{J}$ is known over the cylinder. The reaction between the unknown currents $\underline{J}$ and any assumed distribution of electric currents $\underline{J}^x$ can therefore be calculated. It is precisely the negative of the reaction between $\underline{M}$ and $\underline{J}^x$, that is,

$$\langle x, j \rangle = - \langle x, m \rangle \qquad (IV-6)$$

for an assumed electric current over the antenna boundary.

Now consider the question of how to represent the current along the conductor. One suggestion would be to express the current in a Fourier series, and then determine the coefficients by the reaction approach. This would again involve an infinite set of simultaneous equations in an infinite number of unknowns, being of little practical value. A finite number of terms of the series might be used for an approximate solution, but again a sufficient number of terms would entail a prohibitive amount of work. It is desired to approximate the current with as few functions with unknown coefficients as possible. This can be done by expressing the current in terms of the modes of a cylindrical conductor. Stratton shows that the only symmetric mode external to a perfectly conducting cylinder travels unattenuated with the speed of light.[15] There will be a discontinuity in the current at the gap, but if the length of the gap is small, the discontinuity can be taken at z = 0. From symmetry conditions, the current must be an even function of z. Thus, for a solid conductor, neglecting the currents over the ends of the conductor, the current will be of the form

$$J_z(z) = Ae^{-j\beta|z|} + Be^{j\beta|z|} . \qquad (IV-7)$$

This involves two adjustable constants, A and B. For a hollow cylinder, in addition to the external modes, there will also be a set of internal modes, corresponding to the symmetric TM circular waveguide modes. For small diameter antennas, these modes are all beyond cut-off, and rapidly attenuated. There is also no discontinuity in the internal modes at z = 0, since the boundary conditions are continuous in that region. Thus, for·a hollow cylinder,

$$J_z(z) = Ae^{-j\beta|z|} + Be^{j\beta|z|} + \sum_{n=1}^{\infty} C_n \cosh \alpha_n z , \qquad (IV-8)$$

where the α_n are the propagation constants for the TM_{on} waveguide modes. The internal modes are appreciable only in the immediate vicinity of the ends of the conductor, but it is necessary to include at least one of them in the reaction formulation of the problem.to satisfy the condition J_z ($\pm$b) = 0. Then the approximation (IV-8) again reduces to one involving two adjustable constants.

The reaction formulation for $\langle m,j \rangle$ is now derived as follows. The current distribution is represented by

$$\underline{J} \approx \underline{J}^a = A\underline{J}^v + B\underline{J}^v , \qquad (IV-9)$$

where $\underline{J}^a$ denotes an approximate current on the cylinder consisting of the two partial currents $\underline{J}^u$ and $\underline{J}^v$, with adjustable constants A and B. One can enforce the conditions

$$\langle u,a \rangle = \langle u,j \rangle = - \langle u,m \rangle , \qquad (IV-10)$$

$$\langle v,a \rangle = \langle v,j \rangle = - \langle v,m \rangle , \qquad (IV-11)$$

that is, $\underline{J}^u$ and $\underline{J}^v$ look the same to $\underline{J}^a$ as they do to $\underline{J}$. Eqs. (IV-10) and (IV-11) involve only free-space reactions between known sources, and can thus be calculated. Substituting for $\underline{J}^a$ from Eq. (IV-9) gives the pair of equations

$$A\langle u,u \rangle + B\langle u,v \rangle = - \langle u,m \rangle , \qquad (IV-12)$$

$$A\langle v,u \rangle + B\langle v,v \rangle = - \langle v,m \rangle . \qquad (IV-13)$$

By reciprocity, $\langle u,v \rangle = \langle v,u \rangle$. Solving for A and B by Cramer's rule, one has

$$A = \frac{\langle u,v \rangle \langle v,m \rangle - \langle v,v \rangle \langle u,m \rangle}{\langle u,u \rangle \langle v,v \rangle - \langle u,v \rangle^2} , \qquad (IV-14)$$

$$B = \frac{\langle u,v \rangle \langle u,m \rangle - \langle u,u \rangle \langle v,m \rangle}{\langle u,u \rangle \langle v,v \rangle - \langle u,v \rangle^2} . \qquad (IV-15)$$

The reaction of interest is now

$$\langle m,j \rangle = \langle j,m \rangle \approx \langle a,m \rangle = A\langle u,m \rangle + B\langle v,m \rangle$$

$$= \frac{\langle v,v \rangle \langle u,m \rangle^2 - 2\langle u,v \rangle \langle u,m \rangle \langle v,m \rangle + \langle u,u \rangle \langle v,m \rangle^2}{\langle u,v \rangle^2 - \langle u,u \rangle \langle v,v \rangle} \, . \tag{IV-16}$$

This is the best approximation to $\langle m,j \rangle$ using two adjustable constants. Substituting Eq. (IV-16) into Eq. (IV-5) gives the desired variational expression for Y_c.

V. EVALUATION OF THE REACTIONS

Appendix A gives general expressions for the field from a distribution of z-directed electric currents and ϕ-directed magnetic currents. Although all currents are considered to be on the cylinder $\rho = a$, the magnetic currents are taken to be an infinitesimal distance outside of the magnetic currents. From Eq. (I-16), the reaction between the magnetic currents and any source of interest is

$$\langle m,x \rangle = -2\pi a \int_{-\infty}^{\infty} M_\phi(z) \, H_\phi^x(a,z) \, dz. \tag{V-1}$$

If H_ϕ^x is the magnetic field from a distribution J_z^x on the cylinder, one must take the value of H_ϕ just outside of the cylinder since there is a discontinuity at $\rho = a$. Making use of Parseval's theorem, Eq. (I-15), one can express Eq. (V-1) as

$$\langle m,x \rangle = -a \int_{-\infty}^{\infty} \bar{M}_\phi(-w) \, \bar{H}_\phi^x(a,w) \, dw, \tag{V-2}$$

where $\bar{M}$ and $\bar{H}$ are the transforms of M and H. From Appendix A, one has for the source M_ϕ,

$$\bar{H}_\phi^m(a,w) = -\frac{\pi a \omega \epsilon}{2} \bar{M}_\phi(w) \, H_1^{(2)} \left(a\sqrt{\beta^2 - w^2}\right) J_1\left(a\sqrt{\beta^2 - w^2}\right), \tag{V-3}$$

and for a source J_z^x,

$$\overline{H}_\phi^x(a,w) = -\frac{j\pi a}{2}\overline{J}_z^x(w)(\beta^2-w^2)\ H_1^{(2)}(a\sqrt{\beta^2-w^2}\ )\ J_0(a\sqrt{\beta^2-w^2}\ ). \quad (V-4)$$

First, consider the self-reaction of the magnetic currents. The general expression for this, from Eqs. (V-2) and (V-3), is

$$\langle m,m\rangle = \frac{\omega\epsilon\pi a^2}{2}\int_{-\infty}^{\infty}\overline{M}_\phi(w)\ \overline{M}_\phi(-w)\ H_1^{(2)}(a\sqrt{\beta^2-w^2}\ )\ J_1(a\sqrt{\beta^2-w^2}\ )\ dw. \quad (V-5)$$

Substituting for $\overline{M}$ from Eq. (III-1), one has for Y_g, Eq. (V-4),

$$Y_g = \frac{\omega\epsilon\pi a^2}{2}\int_{-\infty}^{\infty}\left[\frac{\sin wc}{wc}\right]^2 H_1^{(2)}(a\sqrt{\beta^2-w^2}\ )\ J_1(a\sqrt{\beta^2-w^2}\ )\ dw \quad (V-6)$$

This is valid for any size cylinder. For a thin antenna, one can use the approximate formulas for Bessel functions of small argument in Eq. (V-6). This gives

$$Y_g \approx \frac{j\omega\epsilon a^2}{2}\int_{-\infty}^{\infty}\left[\frac{\sin wc}{wc}\right]^2 dw = j\omega\epsilon\ \frac{\pi a^2}{2c}\ . \quad (V-7)$$

It is noted that this is just the capacitive susceptance of a capacitor having the dimensions of the gap. This result is to be expected since circuit concepts apply if the gap dimensions are small compared to wavelength.

Now consider the reactions $\langle m,x\rangle$ for a current J_z^x along the cylinder. From Eqs. (V-2) and (V-4), one has in general

$$\langle x,m\rangle = -\frac{j\pi a^2 V}{2}\int_{-\infty}^{\infty}\overline{J}_z^x(-w)\ \frac{\sin wc}{wc}\ (\beta^2-w^2)\ H_1^{(2)}(a\sqrt{\beta^2-w^2}\ )\ J_0(a\sqrt{\beta^2-w^2}\ )\ dw. \quad (V-8)$$

Again this reduces to a simple expression if the gap dimensions are small. If $\overline{J}_z^x(w) = O(w^{-\alpha})$, $\alpha > 1$, as $w \to \infty$, one can use the approximations for small argument Bessel functions and let $\sin wc/wc \approx 1$. This gives

$$\langle x,m\rangle = aV\int_{-\infty}^{\infty}\overline{J}_z^x(-w)\ dw = 2\pi a V\ J_z^x(0)\ . \quad (V-9)$$

This is recognized as the reaction for a circuit source, VI, which is to be expected if the gap dimensions are small.

For the reactions between electric currents, if the conductor is taken to be hollow, one has from Eq. (I-16),

$$\langle x,y \rangle = 2\pi a \int_{-\infty}^{\infty} J_z^x(z) \ E_z^y(a,z) \ dz, \qquad \text{(V-10)}$$

where J_z^x and J_z^y are any two distributions of electric current along the cylinder $\rho = a$. Again using Parseval's theorem, Eq. (I-15), one has

$$\langle x,y \rangle = a \int_{-\infty}^{\infty} \overline{J}_z^x(-w) \ \overline{E}_z^y(a,w) \ dw. \qquad \text{(V-11)}$$

From Appendix A, the transform of the electric field from a distribution of electric current is

$$\overline{E}_z^y(a,w) = -\frac{\pi a}{2\omega\epsilon} \ \overline{J}_z^y(w) \ (\beta^2-w^2) \ H_o^{(2)}(a\sqrt{\beta^2-w^2}) \ J_o(a\sqrt{\beta^2-w^2}). \qquad \text{(V-12)}$$

Thus, the general formula for the reactions involving electric currents is

$$\langle x,y \rangle = -\frac{\pi a^2}{2\omega\epsilon} \int_{-\infty}^{\infty} \overline{J}_z^x(-w) \ \overline{J}_z^y(w)(\beta^2-w^2) \ H_o^{(2)}(a\sqrt{\beta^2-w^2}) \ J_o(a\sqrt{\beta^2-w^2}) \ dw. \qquad \text{(V-13)}$$

As shown in Section IV, the current along the cylinder should be of the form of Eq. (IV-8). The functional form of the current chosen for the reaction approach must go to zero at $z = \pm b$, else the self-reaction as given by Eq. (V-13) becomes infinite. This is to be expected since it would require an infinite E_z to support a finite J_z at the ends of a hollow cylinder. A choice which consists of the two external modes plus the dominant internal mode is

$$J_z^u(z) = \sin \beta(b - |z|), \qquad \text{(V-14)}$$

$$J_z^v(z) = \cos \beta(b - |z|) - \frac{\cosh \gamma z}{\cosh \alpha b}, \qquad \text{(V-15)}$$

where α is the propagation constant for the TM_{01} mode. For small cylinders, α is very large and

$$J_z^v(0) = \cos \beta b, \tag{V-16}$$

for any antenna of appreciable length. Defining the integrals

$$\langle x,y \rangle = - (2\pi a)^2 \sqrt{\mu/\epsilon} \; I_{xy} , \tag{V-17}$$

one has from Eq. (V-13),

$$I_{uu} = \frac{1}{8\pi\beta} \int_{-\infty}^{\infty} \left[\overline{J}_z^u(w)\right]^2 (\beta^2-w^2) \; H_0^{(2)}(a\sqrt{\beta^2-w^2}) \; J_0(a\sqrt{\beta^2-w^2}) \; dw, \tag{V-18}$$

$$I_{uv} = \frac{1}{8\pi\beta} \int_{-\infty}^{\infty} \overline{J}_z^u(w) \; \overline{J}_z^v(w)(\beta^2-w^2) \; H_0^{(2)}(a\sqrt{\beta^2-w^2}) \; J_0(a\sqrt{\beta^2-w^2}) \; dw, \tag{V-19}$$

$$I_{vv} = \frac{1}{8\pi\beta} \int_{-\infty}^{\infty} \left[J_z^v(w)\right]^2 (\beta^2-w^2) \; H_0^{(2)}(a\sqrt{\beta^2-w^2}) \; J_0(a\sqrt{\beta^2-w^2}) \; dw. \tag{V-20}$$

These simplify to some extent for small diameter antennas, but the details are complicated and outlined in Appendix B. The real parts of the I_{xy} integrals are independent of a if the antenna is thin.

Now substituting the various results of this section into Eqs. (IV-16) and (IV-5), one has for the conductor admittance

$$Y_c = \sqrt{\epsilon/\mu} \; \frac{I_{uu}\cos^2\beta b + I_{vv}\sin^2\beta b - I_{uv}\sin 2\beta b}{I_{uu}I_{vv} - I_{uv}^2} . \tag{V-21}$$

The complete expression for the input admittance to the antenna is given by adding Y_g, Eq. (V-7), to Y_c. Considering the I_{xy} integrals, it is seen that for small βa, I_{vv} becomes very large compared with I_{uv} and I_{uu}. Thus, except in the region of small $\sin \beta b$ (corresponding to the region of resonance), the conductor admittance is given by

$$Y_c \approx \sqrt{\epsilon/\mu} \; \frac{\sin^2\beta b}{I_{uu}} . \tag{V-22}$$

This is precisely the result that is obtained from the reaction approach using one adjustable constant, and approximating the current by Eq. (V-14).

VI. DISCUSSION

The evaluation of the I_{xy} in the reaction or variational solution
to the cylindrical antenna is a difficult task. As shown in Appendix B,
the real parts are for all practical purposes independent of the antenna
diameter, and are fairly simple to evaluate. However, the imaginary
parts are complicated functions of both βa and βb, and no simple expressions
have been obtained for them. Evaluation by numerical integration is possible,
but would be an arduous job.

A comparison of the solution presented in this report to previous
solutions can be made. Consider Storer's variational solution.[3] This
solution used the same approximate integral equation as used in King's
solution. In Appendix C, it is shown that Storer's solution can be put
into the same form as Eq. (V-21), except that the I_{xy} are slightly different.
They are given by Eqs. (V-18) through (V-20) with $J_o(a\sqrt{\beta^2-w^2})$ replaced
by unity. This difference would show up only in the imaginary part of I_{vv},
since the integrands of the other two integrals become negligible before
J_o differs appreciably from unity. It is difficult to predict quantitatively
what the difference in the two solutions would be, but any difference
would be evidenced only in the vicinity of resonance. The source of this
difference is due to Storer's approximation of the integrand of the starting
equation, which is inaccurate in the vicinity of the ends of the conductor.
Thus, the "end effects" are obscured, and it is no longer necessary to
require $J_z^x(z)$ to be zero at $z = \pm b$. Instead of assuming Eq. (V-15) one
could now take $J_z^v(z) = \cos \beta(b - |z|)$, without affecting the evaluation of
I_{xy}. Storer gives a comparison of his solution to that of King, and shows
that the two agree reasonably well.

There are some experimental values of antenna impedance in the region
of resonance available which indicate that both Storer's and King's solu-

tion are reasonably accurate.[17] Before attempting numerical calculations for the solution of this paper, it is hoped that a simpler method of evaluating the imaginary parts of I_{xy} can be devised.

VII. REFERENCES

1. E. Hallén, "Theoretical Investigations into the Transmitting and Receiving Qualities of Antennae," _Nova Acta_, Upsala, Vol. 11, No. 4, 1938.

2. King and Middleton, "The Cylindrical Antenna; Current and Impedance," _Quart. Appl. Math._, Vol. 1, Oct. 1943, pg. 201.

3. J. E. Storer, "Variational Solution to the Problem of the Symmetrical Cylindrical Antenna," Cruft Lab. Report TR101, Harvard Univ., 1950.

4. Schelkunoff and Friis, _Antennas, Theory and Practice_, John Wiley and Sons, Inc., New York, 1952, Chapt. 13.

5. V. H. Rumsey, "The Reaction Concept in Electromagnetic Theory," Antenna Lab. Report 510-6, The Ohio State Univ., 1953.

6. LePage and Seely, _General Network Analysis_, McGraw-Hill Book Co., New York, 1952, pg. 11.

7. Schelkunoff, _Electromagnetic Waves_, D. Van Nostrand Co., New York, 1943, Chapt. 6.

8. Stratton, _Electromagnetic Theory_, McGraw-Hill Book Co., New York, 1941, Chapt. VI.

9. Schelkunoff, _op. cit._, pg. 153.

10. Titchmarsh, _Theory of Fourier_ Integrals, Oxford Press, 1937.

11. Harrington, "On Transform Solutions to Integral Equations Arising From Boundary Value Problems," Antenna Lab. Report, The Ohio State Univ., Sept. 1952.

12. Schelkunoff, _op. cit._, pg. 476.

13. Schelkunoff, <u>Advanced Antenna Theory</u>, John Wiley and Sons, New York, 1952, pg. 149.

14. Sommerfield, <u>Partial Differential Equations in Physics</u>, Academic Press, Inc., New York, pg. 29.

15. Stratton, <u>op. cit.</u>, pp. 527-531.

16. E. Jordan, <u>Electromagnetic Waves and Radiating Systems</u>, Prentice Hall, New York, 1950, pp. 343-347.

17. R. W. P. King, "Electromagnetic Engineering," Vol. II, unpublished notes.

APPENDIX A

The Field from a Cylinder of Currents

It was shown in Section II that the problem of the cylindrical antenna can be reduced to a cylinder of impressed currents radiating into free-space. It was also concluded from symmetry considerations that the magnetic currents flow only in the ϕ direction and the electric currents only in the z direction (if the conductor is taken to be hollow). Such a configuration gives rise to a field that is TM to the z direction.

A circularly symmetric TM field in a homogeneous, source-free region can be expressed in terms of an $\underline{A}$ potential having only a z component given by[8]

$$A_z(\rho,z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(w) \, Z_0\left(\rho\sqrt{\beta^2 - w^2}\right) e^{-jwz} \, dz. \tag{A-1}$$

The function Z_0 is a solution to Bessel's equation, being J_0 if the axis $\rho = 0$ is included, and $H_0^{(2)}$ if the region $\rho \rightarrow \infty$ is included. The function $f(w)$ is to be determined from boundary conditions.

Consider now a distribution of ϕ-directed magnetic currents and z-directed electric currents on a given cylinder $\rho = a$. Let M_ϕ denote the surface density of the magnetic current, and J_z denote the surface density of the electric current. Let a superscript e denote a quantity external to the cylinder, and i denote one internal. The boundary conditions at the given cylinder are

$$J_z(z) = H_\phi^e(a,z) - H_\phi^i(a,z) , \tag{A-2}$$

$$M_\phi(z) = E_z^e(a,z) - E_z^i(a,z) . \tag{A-3}$$

The field components of interest are

$$H_\phi(\rho,z) = -\frac{\partial}{\partial\rho} A_z(\rho,z) , \tag{A-4}$$

$$j\omega\epsilon E_z(\rho,z) = (\beta^2 + \frac{\partial^2}{\partial z^2}) A_z(\rho,z) \; , \tag{A-5}$$

with A_z as given by Eq. (A-1).

It can be shown that the Fourier transforms of the field vectors exist if the sources can be contained in a finite volume,[11] or, more generally, if the sources are transformable. Since the Fourier transform is unique,

$$A_z(\rho,w) = f(w) \; Z_o(\rho\sqrt{\beta^2-w^2}) \tag{A-6}$$

is the transform of $A_z(\rho,z)$. From Eqs. (A-4) and (A-5), it follows that

$$\bar{H}_\phi(\rho,w) = -\frac{\partial}{\partial\rho} \bar{A}_z(\rho,w) \; , \tag{A-7}$$

$$j\omega\epsilon\bar{E}_z(\rho,w) = (\beta^2-w^2) \; \bar{A}_z(\rho,w) \; , \tag{A-8}$$

where $\bar{E}_z$ and $\bar{H}_\phi$ denote the transforms of E_z and H_ϕ. From Eqs. (A-2) and (A-3), it follows that

$$\bar{J}_z(w) = \bar{H}_\phi^e(a,w) - \bar{H}_\phi^i(a,w) \; , \tag{A-9}$$

$$\bar{M}_\phi(w) = \bar{E}_z^e(a,w) - \bar{E}_z^i(a,w) \; , \tag{A-10}$$

where $\bar{J}_z$ and $\bar{M}_\phi$ are the transforms of J_z and M_ϕ.

Now the expressions for the quantities of interest can be written out explicitly for the two regions. Denoting the unknown f for the internal region by $f^i(w)$, one has

$$\bar{A}_z^i(\rho,w) = f^i(w) \; J_o(\rho\sqrt{\beta^2-w^2}) \; , \tag{A-11}$$

$$\bar{E}_z^i(\rho,w) = \frac{(\beta^2-w^2)}{j\omega\epsilon} \; f^i(w) \; J_o(\rho\sqrt{\beta^2-w^2}) \; , \tag{A-12}$$

$$\bar{H}_\phi^i(\rho,w) = \sqrt{\beta^2-w^2} \; f^i(w) \; J_1(\rho\sqrt{\beta^2-w^2}) \; . \tag{A-13}$$

Denoting f for the external region by $f^e(w)$ one has

$$\bar{A}_z^e(\rho,w) = f^e(w) \; H_o^{(2)}(\rho\sqrt{\beta^2-w^2}) \; , \tag{A-14}$$

$$\bar{E}_z^e(\rho, w) = \frac{(\beta^2 - w^2)}{jw\epsilon} f^e(w) H_o^{(2)}(\rho\sqrt{\beta^2 - w^2}) , \qquad (A-15)$$

$$\bar{H}_\phi^e(\rho, w) = \sqrt{\beta^2 - w^2} f^e(w) H_1^{(2)}(\rho\sqrt{\beta^2 - w^2}) . \qquad (A-16)$$

One can now determine the f functions from the boundary conditions at $\rho = a$. Substituting into Eqs. (A-9) and (A-10) gives a pair of equations

$$f^e(w) H_o^{(2)}(a\sqrt{\beta^2 - w^2}) - f^1(w) J_o(a\sqrt{\beta^2 - w^2}) = \frac{jw\epsilon}{\beta^2 - w^2} \bar{M}_\phi(w) , \qquad (A-17)$$

$$f^e(w) H_1^{(2)}(a\sqrt{\beta^2 - w^2}) - f^1(w) J_1(a\sqrt{\beta^2 - w^2}) = \frac{1}{\sqrt{\beta^2 - w^2}} J_z(w) . \qquad (A-18)$$

The determinant of the coefficients of the f functions is recognized as j times the Wronskian of Bessel's equation, that is

$$\begin{vmatrix} H_o^{(2)}(a\sqrt{\beta^2 - w^2}) & -J_o(a\sqrt{\beta^2 - w^2}) \\ H_1^{(2)}(a\sqrt{\beta^2 - w^2}) & -J_1(a\sqrt{\beta^2 - w^2}) \end{vmatrix} = \frac{j \, 2}{a\sqrt{\beta^2 - w^2}} \qquad (A-19)$$

Solving Eqs. (A-16) and (A-17) by Cramer's rule gives

$$f^1(w) = \frac{\pi a}{2j}\left[\frac{-jw\epsilon}{\sqrt{\beta^2 - w^2}} H_1^{(2)}(a\sqrt{\beta^2 - w^2}) \bar{M}_\phi(w) + H_o^{(2)}(a\sqrt{\beta^2 - w^2}) \bar{J}_z(w)\right] \quad (A-20)$$

$$f^e(w) = \frac{\pi a}{2j}\left[\frac{-jw\epsilon}{\sqrt{\beta^2 - w^2}} J_1(a\sqrt{\beta^2 - w^2}) \bar{M}_\phi(w) + J_o(a\sqrt{\beta^2 - w^2}) \bar{J}_z(w)\right] . \quad (A-21)$$

The electromagnetic field is now determined at all points in space. Explicitly, it is given by substituting Eqs. (A-20) and (A-21) into Eqs. (A-11) through (A-16), and applying the inversion integral.

APPENDIX B

On Evaluating the I_{xy}

Taking the transforms of the assumed currents, Eqs. (V-14) and (V-15), one has

$$\bar{J}_z^u(w) = \frac{2\beta(\cos wb - \cos \beta b)}{\beta^2 - w^2} , \tag{B-1}$$

$$\bar{J}_z^v(w) = \frac{2(\beta \sin \beta b - w \sin wb)}{\beta^2 - w^2} - \frac{2(w \sin wb + \alpha \tanh \alpha b \cos wb)}{\alpha^2 + w^2}$$

$$\approx \frac{2(\beta \sin \beta b - w \sin wb)}{\beta^2 - w^2} - \frac{2(w \sin wb + \alpha \cos wb)}{\alpha^2 + w^2} \tag{B-2}$$

The approximation in the second expression for $\bar{J}_z^v$ is permissible if α is large, that is, if the antenna is thin. The integrals of interest, from Eqs. (V-18) through (V-20), are

$$I_{xy} = \frac{1}{8\pi\beta} \int_{-\infty}^{\infty} \bar{J}_z^x(w) \, \bar{J}_z^y(w) \, (\beta^2-w^2) \, H_0^{(2)}(a\sqrt{\beta^2-w^2}) \, J_0(a\sqrt{\beta^2-w^2}) dw. \tag{B-3}$$

All functions in the integrand are real except $H_0^{(2)}$. This can be separated into its real and imaginary parts as

$$\text{Re}\left[H_0^{(2)}(a\sqrt{\beta^2-w^2})\right] = J_0(a\sqrt{\beta^2-w^2}) \quad , \quad |w| < \beta ,$$

$$\text{Im}\left|H_0^{(2)}(a\sqrt{\beta^2-w^2})\right| = -N_0(a\sqrt{\beta^2-w^2}) \quad , \quad |w| < \beta , \tag{B-4}$$

$$= \frac{2}{\pi} K_0(a\sqrt{w^2-\beta^2}) \quad , \quad |w| > \beta .$$

The integrand of Eq. (B-3) is an even function of w. Using Eqs. (B-4), one has

$$\text{Re}(I_{xy}) = \frac{1}{4\pi\beta} \int_0^\beta \bar{J}_z^x(w) \, \bar{J}_z^y(w) \, (\beta^2-w^2) \, J_0^2(a\sqrt{\beta^2-w^2}) \, dw \tag{B-5}$$

$$\text{Im}(I_{xy}) = \frac{-1}{4\pi\beta} \int_0^\beta \bar{J}_z^x(w) \, \bar{J}_z^y(w) \, (\beta^2-w^2) \, N_0(a\sqrt{\beta^2-w^2}) \, J_0(a\sqrt{\beta^2-w^2}) \, dw$$

$$+ \frac{1}{2\pi^2\beta} \int_\beta^\infty \bar{J}_z^x(w) \, \bar{J}_z^y(w) \, (\beta^2-w^2) \, K_0(a\sqrt{w^2-\beta^2}) \, I_0(a\sqrt{w^2-\beta^2}) \, dw . \tag{B-6}$$

It is now noted that for thin antennas, $J_0(a\sqrt{\beta^2-w^2}) \approx 1$ for $|w| < \beta$. Letting $\beta w = u$, the real parts of the I_{xy} reduce to particularly simple forms. Noting that the second term of Eq. (B-2) is negligible compared to the first term for $|w| < \beta$, one has

$$\text{Re}(I_{uu}) = \frac{1}{\pi} \int_0^1 \frac{(\cos u\beta b - \cos \beta b)^2}{1 - u^2} \, du \, , \qquad (B-7)$$

$$\text{Re}(I_{uv}) = \frac{1}{\pi} \int_0^1 \frac{(\cos u\beta b - \cos \beta b)(\sin \beta b - u \sin u\beta b)}{1 - u^2} \, du \, , \qquad (B-8)$$

$$\text{Re}(I_{vv}) = \frac{1}{\pi} \int_0^1 \frac{(\sin \beta b - u \sin u\beta b)^2}{1 - u^2} \, du \, . \qquad (B-9)$$

These can be readily evaluated in terms of sine and cosine integrals or by numerical integration.

Unfortunately, the imaginary parts of the I_{xy} have not been reduced to simple form. They can be evaluated using various approximations for the Bessel functions. The resulting formulas are quite complex, and will not be given in this report.

APPENDIX C

Alternate Derivation of the Reactions

Instead of solving for the field from a cylinder of currents as a boundary value problem, as was done in Appendix A, one can use the potential integral formulation.[7] For a cylinder of z-directed electric currents, E_z is given by

$$E_z^x(\rho,z) = \frac{a}{4\pi j\omega\epsilon} (\beta^2 + \frac{\partial^2}{\partial z^2}) \int_0^{2\pi} d\phi' \int_{-b}^{b} dz' \, J_z^x(z') \frac{e^{-j\beta r''}}{r''} , \qquad (C-1)$$

where

$$r'' = \sqrt{a^2 + \rho^2 - 2\rho a \cos(\phi-\phi') + (z-z')^2} . \qquad (C-2)$$

Using the approximation employed by Hallén, King, and Storer,[3] Eq. (C-1) for $\rho = a$ becomes

$$E_z^x(a,z) = \frac{a}{2j\omega\epsilon} (\beta^2 + \frac{\partial^2}{\partial z^2}) \int_{-b}^{b} J_z^x(z') \frac{e^{-j\beta\sqrt{(z-z')^2+a^2}}}{\sqrt{(z-z')^2+a^2}} dz' . \qquad (C-3)$$

This equation is accurate except in the vicinity of $z = -b$. It thus effectively masks the "end effects" of the antenna. Making use of the known integral,[11]

$$\int_{-\infty}^{\infty} \frac{e^{-j\beta\sqrt{(z-z')^2+a^2}}}{\sqrt{(z-z')^2+a^2}} e^{-jwz} dz = \frac{\pi}{j} e^{-jwz'} H_0^{(2)}(a\sqrt{\beta^2-w^2}) , \qquad (C-4)$$

one can take the transform of Eq. (C-4), giving

$$\bar{E}_z^x(w) = - \frac{\pi a}{2\omega\epsilon} \bar{J}_z^x(w) (\beta^2-w^2) H_0^{(2)}(a\sqrt{\beta^2-w^2}) . \qquad (C-5)$$

It is noted that this is the same as obtained for the exact formulation, Eq. (V-12), except that the term $J_0(a\sqrt{\beta^2-w^2})$ is replaced by unity. This term is approximately unity for small βa and small w, giving a discrepancy between Eqs. (C-5) and (V-12) only for large values of w. This is to be expected since the behavior of $\bar{E}_z^x(w)$ as $|w|\to\infty$ is determined by the behavior of $E_z^x(a,z)$ as $z\to\pm b$, and this effect was

masked in the derivation of Eq. (C-5). Since $\overline{E}_z^x(w)$ goes to zero as $e^{-\beta a|w|}$ as $w \to \pm\infty$, it is no longer necessary to require $J_z^x(\pm b) = 0$. Thus, instead of Eq. (V-15), one could take

$$J_z^v(z) = \cos \beta(b - |z|) , \tag{C-6}$$

with $J_z^u(z)$ as given by Eq. (V-14). The transform of J_z^v now becomes

$$\overline{J}_z^v(w) = \frac{2(\beta \sin \beta b - w \sin wb)}{\beta^2 - w^2} , \tag{C-7}$$

and the transform of J_z^u remains as given by Eq. (B-1). The integrals appearing in Eq. (V-21) become

$$I_{xy} = \frac{1}{8\pi\beta} \int_{-\infty}^{\infty} \overline{J}_z^x(w) \, \overline{J}_z^y(w) \, (\beta^2 - w^2) \, H_o^{(2)}(a\sqrt{\beta^2 - w^2}) \, dw \tag{C-8}$$

for the formulation of this appendix. A comparison of the I_{xy} as given by Eq. (C-8) with those given by Eq. (B-3) shows that only the imaginary part of I_{vv} might differ appreciably for the two formulations. A quantitative comparison cannot be made until numerical evaluation is attempted.